Self-fulfilling Prophecies in Directed Technical Change*

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Abstract

Directed technical change (DTC) may stimulate clean innovations and phase out polluting technologies, thus contributing to climate change policy. In the existing literature on DTC, market size and initial conditions determine whether the direction of technical change is clean or dirty and path dependency arises (see for example Acemoglu et al., 2012). However, as the literature on coordination failures has pointed out in a different context, expectations play an important role in forward-looking decision makings (see for example Krugman, 1991; Matsuyama, 1991). This paper shows how multiple equilibria easily arises in a standard workhorse model of DTC when innovators are forward-looking. In our model there is a range of initial conditions from which both an equilibrium with clean innovation and an equilibrium with dirty innovation can emerge. Accordingly, the transition to an economy dominated by green technologies is a self-fulfilling prophecy, but the transition to an economy that is locked in brown technologies is a self-fulfilling prophecy as well. The range for which this multiplicity arises is shown to depend on the degree of substitutability of the final goods from the two sectors. We also investigate the implications of the existence of the overlap for environmental policies.

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1 Introduction

While the climate negotiation process is often seen as a long process without much progress so far, sometimes sparkles of hope light up, as when the Paris 2015 COP meeting created some optimistic reactions. Such optimism has continued to fuel the momentum of Paris Agreement, until Syria ratified the agreement late 2017, leaving the United States as the only country on earth opposed, after it withdrew under President Trump. Whether this hope is justified or not is not yet clear, but maybe just the change in sentiment can help already. A parallel can be drawn to the experience with smoking bans, about which it seemed impossible to reach a consensus until suddenly opinions got coordinated among stakeholders.

A big challenge for the implementation of environmental policies comes from vested interests and existing technologies, which act as historically grown barriers that lock-in society into emission-intensive technologies. Companies are reluctant to give up their firm position in polluting industries and even for society at large it may be too costly to forego all the benefits from these sectors and start a green growth path from scratch – the drop in output may be simply too big. However, while ignoring history is too costly, ignoring the future might be even more so, not only for society. If firms realize that the future needs to be green, they might realize that their future is in green products and that they prefer to stop investing in declining industries in order to benefit from the size of the future green markets and from spillovers, which create complementarities between firms investment decisions. Forward-looking investors might find it attractive to go for a business strategy that tends to be green if the whole market is going green. Self-fulfilling prophecies might replace the force of lock-in: if everybody believes the future is in clean production, all investment might move in this direction.

This paper shows that in the most natural macro-economic dynamic setting with investment in clean and polluting technologies, expectations or beliefs about future environmental-friendliness of innovation can overturn the lock-in in polluting technology that resulted from a history of past investment in these technologies. Thus we show that in a model of directed technical change, not only history play a role but also expectations, as shown in a different context by Krugman (1991) and Matsuyama (1991).

Our model is similar to the model by Acemoglu et al. (2012), but gives different results. The main reason is that we allow for investment to be based on expected returns over a long (infinite) horizon, while Acemoglu et al. (2012) assume patents to last for only one period. We find that two stable steady-state equilibria can co-exist and that the selection between the two involves a coordination problem. In addition to this “global indeterminacy” result, we also find “local indeterminacy” (in the terminology of Benhabib and Perli, 1994): multiple transitional paths exist to each of the steady-state equilibria which are consistent with rational deterministic expectations, i.e. expectations that can be rationalized in an equilibrium without stochastic shocks. The implication is that economies
with identical preferences and technologies could have quite different patterns of technical change both in the short and long run.

Multiple equilibria and coordination problems have been studied in endogenous growth theory literature. Examples relying on the increasing returns and/or externalities to generate indeterminacy include Benhabib and Perli (1994), Boldrin and Rustichini (1994) and Benhabib et al. (2008), while rational-expectation-based indeterminacy can be found in Cozzi (2005), Cozzi (2007) and Gil (2013). In models with a focus on environmental issues, however, multiple equilibria is often associated with the discussion of tipping point and Skiba point (Skiba, 1978) as in the literature of natural resource management, such as in Maeler et al. (2003), or related to path dependency and lock-in phenomenon in models dealing with directed technical change, such as Acemoglu et al. (2012). By emphasizing the role of initial conditions, these models typically ignore the role of indeterminacy – the fact that multiple, saddle-point stable equilibria exist and are each consistent with rational expectations – with Wirl (2004), Van der Meijden and Smulders (2017) and Bretschger and Schaefer (2017) among the few exceptions.

Although certainly a very important factor to consider, path dependency might not be the entire story that characterizes directed technical change. As the discussions about “animal spirits” (Howitt and McAfee, 1988) or “the waves of enthusiasm” (Cozzi, 2005) suggest, multiple equilibria, sunspot beliefs (Cass and Shell, 1983) or self-fulfilling expectations (Krugman, 1991; Matsuyama, 1991) could be a major driving force as well. In particular, the combination of directed technical change and technology spillovers typically gives rise to “strategic complementarities” and increasing returns, which in turn are well-known ingredients of multiple equilibria and indeterminacy (see also Benhabib and Farmer, 1999).

The rest of the paper is organized as follows: Section 2 presents a modified version of the Acemoglu et al. (2012) model. Section 3 studies its dynamics and proves the global and local indeterminacy; in particular, the rich patterns of expectations will be discussed. A numerical example of the equilibrium paths is provided in Section 4. In Section 5, three different policy measures and their respective impact on the transition are discussed. And finally, Section 6 concludes.

2 The Model

We adopt the Acemoglu et al. (2012) model in a continuous time setting with two major modifications. Contrary to the original model, we, first, allow labor to be mobile between production and research and, second, adopt an infinite patent length instead of a one-period patent length. In the following, the time subscript is omitted whenever no confusion would arise.
2.1 Final Goods Producers

There are two final goods sectors in the economy: clean \((Y_c)\) and dirty \((Y_d)\), differentiated by the fact that the intermediate goods used in the dirty sector causes pollution. The final goods are produced using labor and a continuum of sector-specific intermediates following a Cobb-Douglas technology:

\[
Y_j = L_j^{1-\alpha} \int_0^1 q_{ji} x_{ji}^\alpha di,
\]

where \(j \in \{c, d\}\) denotes the sector, clean or dirty, where \(L_j\) is the production labor hired in sector \(j\), and where \(q_{ji}\) and \(x_{ji}\) are the quality and quantity, respectively, of the intermediate good \(i\) in sector \(j\). The final goods producers are price takers in both the final goods market and the factor markets, and maximize their profit according to

\[
\max_{\{L_j, x_{ji}\}} \pi^F_j = P_j Y_j - wL_j - \int_0^1 P_{ji} x_{ji} di,
\]

where \(P_j\) and \(P_{ji}\) are the output and intermediate input prices, respectively, while \(w\) is the wage. Profit maximization leads to the following factor demand functions:

\[
w = (1 - \alpha) P_j \frac{Y_j}{L_j}, \quad (1)
\]

\[
P_{ji} = \alpha P_j L_j^{1-\alpha} q_{ji} x_{ji}^{\alpha-1} \equiv p(x_{ji}, q_{ji}). \quad (2)
\]

2.2 Intermediate Goods Producers

Each individual intermediate good \(x_{ji}\) is produced by a monopolist. Demand for her product is given by (2). Production requires sector-specific inputs and the unit cost of production increases with the quality of intermediate goods so that one unit of intermediate good requires \(q_{ji}\) units of final output from sector \(j\). Hence, operating profits equal

\[
\pi_{ji} = [p(x_{ji}, q_{ji}) - P_{ji} q_{ji}] x_{ji}.
\]

The intermediate monopolist in addition hires research labor in R&D, \(s_{ji}\), in order to improve the quality of its products.\(^1\) Quality improves in proportion to the firms research input, \(s_{ji}\), spillovers from other firms in the sector as measured by the sector-wide average quality level, \(Q_j\), and the sector-specific productivity parameter \(\mu_j\):

\[
q_{ji} = \mu_j Q_j s_{ji}. \quad (3)
\]

\(^1\)We choose to model innovation as the result of inhouse R&D a la Smulders and Nooij (2003), rather than “creative destruction”. The former gives simpler mathematical expressions and seems to be at least equally empirically relevant as is shown recently by Garcia-Macía et al. (2016). This modeling choice does not alter the qualitative results in this paper as is discussed in Appendix G).
\[ Q_j = \int_0^1 q_{ji} di. \]  

(4)

Due to the spillover term \( Q_j \), the returns on research by one firm also depends on the effort of all other firms in the same sector.

The intermediate goods monopolists choose the amount of production \( x_{ji} \) and the level of research effort \( s_{ji} \) to maximize the net present value of its profits,\(^2\) which leads to the following first order conditions:

\[
P_{ji} = \frac{1}{\alpha} P_j q_{ji},
\]

(5)

\[
\mu_j Q_j \lambda_{ji} \leq w \perp s_{ji} \geq 0,
\]

(6)

\[
\dot{\lambda}_{ji} = r \lambda_{ji} - \frac{\partial \pi_{ji}}{\partial q_{ji}},
\]

(7)

where \( \lambda_{ji} \) is the firm’s shadow price of quality improvements and \( \pi_{ji} \) firm’s profit. Equation (5) is the usual mark-up pricing rule. Equation (6) is the first order condition for the individual monopolist’s investment decision, equating marginal benefits and costs in case of active research. The expression \( \mu_j Q_j \lambda_{ji} \) represents the contribution of a marginal unit of research to the present value of the firm’s future profit (or in other words, the productivity of a marginal unit of research in sector \( j \)), while the wage \( w \) is the marginal cost of research effort. Finally Equation (7) is the arbitrage equation that determines the shadow value of quality improvements as the net present value of future profits due to the quality improvement.

Combining equations we can characterize the production side of the economy as follows (see Appendix A):

\[
Y_j = \alpha^{\frac{2\alpha}{1-\alpha}} Q_j L_j,
\]

(8)

\[
w = (1 - \alpha) \alpha^{\frac{2\alpha}{1-\alpha}} Q_j P_j,
\]

(9)

\[
\dot{Q}_j = \mu_j Q_j s_j,
\]

(10)

\[
x_{ji} = x_j, \quad \lambda_{ji} = \lambda_j,
\]

(11)

\[
x_j = \alpha^{\frac{2}{1-\alpha}} L_j,
\]

(12)

\[
\pi_{ji} = \frac{1 - \alpha}{\alpha} \frac{1}{\alpha^{\frac{2}{1-\alpha}}} L_j P_j q_{ji},
\]

(13)

where \( s_j = \int_0^1 s_{ji} di \) is sectoral research input. Equations (8), (9), and (10) show that the model has a Ricardian production structure in which labor is (both for final goods and intermediate goods) the only primary factor of production of which aggregate productivity is proportional to the aggregate quality stock \( Q \), which grows with aggregate research effort. Because costs and demand of intermediates both increase linearly in the firm’s quality \( q \),

\(^2\)The monopolist maximizes the Hamiltonian \( \max_{x_{ji}, s_{ji}} H_{ji} = \pi_{ji} - w s_{ji} + \lambda_{ji} q_{ji} \) subject to (2) and (3).
all firms in a sector sell the same quantity, \( x_j \), and face the same marginal contribution of quality improvements to profits, \( \partial \pi_{qji} / \partial q_{ji} \), as is shown in (11), (12), and (13). The result is that they all face the same shadow price of quality, \( \lambda_j \), and the same return to investment in quality. Denoting any sector with active research by \( k \), so that \( s_k > 0 \), and substituting (6), (11), (12), and (9) into (7), we find the following equation to characterize the rate of return to innovation (see Appendix A for details):

\[
r = \alpha \mu_k L_k + \hat{w} - \hat{Q}_k,
\]

(14)

where the hats denote growth rates.

### 2.3 The Households

Each point in time, the representative household derives utility from a composite consumption good, \( C \), which is made up by the two substitutable consumption goods, clean \( (C_c) \) and dirty \( (C_d) \), according to a CES instantaneous utility function

\[
C = \left[ C_c^{\frac{\sigma - 1}{\sigma}} + C_d^{\frac{\sigma - 1}{\sigma}} \right]^{\frac{\sigma}{\sigma - 1}},
\]

where \( \sigma \) is the elasticity of substitution between clean and polluting consumption goods. The household also derives utility from the environmental quality, \( E \). To simplify the analysis, we assume separability between \( C \) and \( E \) and a logarithmic specification:

\[
U = \ln C + u(E),
\]

where \( u(\cdot) \) is some concave function.

The household invests in assets, \( V \), with return \( r \). It supplies inelastically one unit of labor at wage \( w \). Subject to the intertemporal budget constraint \( \dot{V} = rV + w - P_cC_c - P_dC_d \), the household maximizes life-time utility \( W_0 = \int_0^\infty [\ln C(t) + u(E(t))]e^{-\rho t}dt \), where \( \rho \) is the utility discount rate. The maximization leads to the usual static demand functions and Euler equation for consumption, respectively:

\[
\frac{C_c}{C_d} = \left( \frac{P_c}{P_d} \right)^{-\sigma},
\]

(15)

\[
r = \rho + \hat{C} + \hat{P},
\]

(16)

where \( P \) is the price index of consumption. Equation (15) shows that relative demand responds to the relative price with elasticity \( \sigma \). Equation (16) shows that households require a rate of return on their savings that reflects their impatience (\( \rho \)), a premium for postponing consumption (\( \hat{C} \)), and for inflation (\( \hat{P} \)).
We define the share of good $j$ in total consumption expenditure as:

$$\theta_j \equiv \frac{P_j C_j}{PC}.$$ 

The price index $P$ is defined by $PC = P_cC_c + P_dC_d$, so that we may write

$$\hat{P} + \hat{C} = \theta_c \left( \hat{P}_c + \hat{C}_c \right) + \theta_d \left( \hat{P}_d + \hat{C}_d \right). \quad (17)$$

### 2.4 Market Equilibrium

Goods market clearing requires that in each sector total production net of intermediates production equals consumption:

$$C_j = (1 - \alpha^2)Y_j. \quad (18)$$

Labor market clearing requires that total (exogenous) supply equals demand for production and research:

$$1 = L + s_c + s_d, \quad (19)$$

where $L$ is total production labor:

$$L = L_c + L_d. \quad (20)$$

The static equilibrium, i.e. equilibrium quantities given the predetermined state variables $Q_c$ and $Q_d$, can be characterized as follows. We denote relative variables by subscript $r$, such that for any variable $z$, we have $z_r \equiv z_c/z_d$. From (18) and (8), we find $C_r = Q_rL_r$, while from (9) we find $P_rQ_r = 1$ thus (15) gives $C_r = Q_r^\sigma$. Together with $\theta_r = P_rC_r$ this implies

$$\frac{L_c}{L_d} = \left( \frac{Q_c}{Q_d} \right)^{\sigma-1} = \frac{\theta_c}{\theta_d}. \quad (21)$$

Since $\theta_c + \theta_d = 1$, we can rewrite this as

$$\theta_j = \frac{L_j}{L} = \frac{Q_j^{\sigma-1}}{Q_c^{\sigma-1} + Q_d^{\sigma-1}}, \quad (22)$$

from which we conclude that a sector’s share in production labor, $L_j/L$, equals its share in consumption expenditure, $\theta_j$, which is pre-determined by the state variables.

We can now characterize the dynamics of the clean sector share by time differentiating (22), which gives:

$$\dot{\theta}_c = \theta_c (1 - \theta_c) (\sigma - 1) (\hat{Q}_c - \hat{Q}_d). \quad (23)$$

The equation implies that if substitution is good ($\sigma > 1$), the clean sector grows relative to the polluting sector as long as clean innovation proceeds relatively fast.
The dynamics are also governed by equilibrium on the capital market. Equation (14) and (16) can be interpreted as the demand and supply, respectively, in the capital market. We first rewrite (16) as

\[ r = \rho + \hat{L} + \hat{w}, \]

where we have used the result that – because of constant mark-ups and constant share of intermediates in sectoral output – final goods consumption expenditure is proportional to wage cost in the production sector, \( PC \propto wL \).

Combining (14), (24), and the first equality in (22), we characterize capital market equilibrium as follows:

\[ \dot{L} = L \left[ \alpha \mu k \theta k - \hat{Q} k - \rho \right] \]

where we recall that sector \( k \) is a research-active sector, i.e. the equation only holds if \( s_k > 0 \). The equation is the general equilibrium version of the Ramsey rule and shows that equilibrium production of consumption goods is more postponed to the future (i.e. \( \dot{L}/L > 0 \)), the more the discount rate falls short of the real rate of return to innovation. The latter is proportional to market size \( \theta k L \) and the productivity of R&D, \( \mu k \), and is reduced by R&D cost reductions as captured by \( \hat{Q} k \).

3 The Analysis

3.1 Innovation Regimes and Steady States

We need to distinguish a few different coordination outcomes, which we call different regimes, depending on which sector – the clean sector, the polluting one, or both – actively innovates. From (6) we see that if \( \mu r Q r \lambda r > 1 \) only the clean sector innovates:\(^4\) in the clean sector marginal benefits and costs of innovation are equalized, but in the polluting sector the marginal benefits fall short of the cost. The condition can be rewritten as:

\[ m c \equiv \frac{Q c \lambda c}{Q c \lambda c + Q d \lambda d} > \frac{1/\mu c}{1/\mu c + 1/\mu d} \equiv \kappa c \iff s c > 0, s d = 0, \]

where \( m c \) represents the share of the clean sector in the total market valuation (recall that \( \lambda j \) is the shadow price of investment in sector \( j \), i.e. the stock price, while \( Q j \) represents total assets in sector \( j \)), and \( \kappa c \) is a composite parameter that represents the relative cost in the clean sector (related to the inverse of the productivity parameters \( \mu j \)). Symmetrically,

\(^3\)More formally, from (8) and (18) we find \( \hat{P} j + \hat{C} j = \hat{P} j + \hat{Q} j + \hat{L} j \) which from (9) equals \( \hat{w} + \hat{L} j \). Substituting this into (17) and (16), and using the definition of \( L \) in (20), we find (24).

\(^4\)We ignore the stagnant case with \( s c = s d = 0 \). In terms of parameters, if \( \mu j \) is relatively large or \( \rho \) relatively low (more precisely, if \( \rho < \alpha \mu j \)), there will always be innovation in equilibrium, and we only need to focus on the cases where innovation is active in at least one of the two sectors. The more restrictive condition in Assumption 1 is needed, on the other hand, to warrant active innovation in both sectors..

\(^5\)Let \( a j \equiv Q j \lambda j \) and \( b j \equiv 1/\mu j \), then \( \mu r Q r \lambda r > 1 \iff a r > b r \iff a r /(1 + a r) > b r /(1 + b r) \). Using the definition \( a r = a c /a d \), we find \( a r /(1 + a r) = a c /(a c + a d) \) and similar for the RHS.
we can characterize the equilibrium with only the polluting sector innovating and the one with both sectors innovating by, respectively:

\[ m_c < \kappa_c \Leftrightarrow s_c = 0, s_d > 0, \quad (27) \]
\[ m_c = \kappa_c \Leftrightarrow s_c > 0, s_d > 0. \quad (28) \]

Intuitively the forward-looking market share of a sector has to be big enough relative to its relative cost of innovation to generate innovation in that sector in equilibrium.

The dynamics of the market valuation share follow directly from its definition in (26) and can be written as

\[ \dot{m}_c = m_c(1 - m_c)(\sigma - 1)\mu_c(1 - L) \]

For future use we define \( m_d = 1 - m_c \) as the polluting sector’s market valuation share.

In light of equations (26), (27), and (28), we distinguish between three regimes to characterize the direction of innovation within a non-degenerate period of time. First, in the clean-only regime, private innovation efforts take place in the clean sector only, \( s_c > 0 \) and \( s_d = 0 \), which requires \( m_c > \kappa_c \). Second, in the dirty-only regime we similarly have \( s_c = 0, s_d > 0, \) and \( m_c < \kappa_c \). Third, the simultaneous regime is characterized by \( s_c > 0, s_d > 0 \). We require this to happen over a non-degenerate period of time so that in the simultaneous regime not only \( m_c = \kappa_c \) is required but also \( \dot{m}_c = 0 \). Notice that \( m_c \) is a continuous variable that cannot jump (unless unexpected shocks arise). Hence, if it starts below \( \kappa_c \) and the economy is in the dirty-only regime, it remains so for some period.

We now characterize each of the three regimes in terms of the three variables \( \theta_c, L, \) and \( m_c \). We do this by using (19) and (10) to eliminate \( s_j \) and \( \hat{Q}_j \), respectively, from equations (23), (25), and (29); and by using equations (6), (7), (11), (12), and (9) to eliminate \( \hat{\lambda}_j \) from equation (29). The result is summarized in the following lemma. The proof and derivation details for all the lemmas and propositions of the paper are provided in the appendix.

**Lemma 1.** The three regimes are characterized as follows.

In the clean-only regime, \( m_c > \kappa_c, s_d = 0, s_c = 1 - L \), and

\[
\begin{align*}
\dot{\theta}_c &= \theta_c(1 - \theta_c)(\sigma - 1)\mu_c(1 - L), \\
\dot{L} &= -\mu_c L \left( 1 + \rho / \mu_c - (1 + \alpha \theta_c) L \right), \\
\dot{m}_c &= \mu_c m_c \left[ \alpha L (m_c - \theta_c) + (1 - L)(1 - m_c) \right].
\end{align*}
\]

In the dirty-only regime, \( m_c < \kappa_c, s_c = 0, s_d = 1 - L \), and
\[
\begin{align*}
\dot{\theta}_c &= -\theta_c(1 - \theta_c)(\sigma - 1)\mu_d(1 - L), \\
\dot{L} &= -\mu_dL\left[1 + \rho/\mu_d - (1 + \alpha(1 - \theta_c))\right], \\
\dot{m}_c &= -\mu_d(1 - m_c)\left[\alpha L (\theta_c - m_c) + (1 - L)m_c\right].
\end{align*}
\] (31)

In the simultaneous regime, \(m_c = \kappa_c, s_c = (1 - \kappa_c)(1 - L) + (\theta_c - (1 - \kappa_c))\alpha L = 1 - L - s_d,\)

\[
\begin{align*}
\dot{\theta}_c &= \theta_c(1 - \theta_c)(\sigma - 1)\alpha L(\mu_c + \mu_d)(\theta_c - \kappa_c), \\
\dot{L} &= (1 - \kappa_c)\mu_cL\left[1 + \alpha L - \left(1 + \rho/(1 - \kappa_c)\mu_c\right)\right], \\
\dot{m}_c &= 0.
\end{align*}
\] (32)

Within each of the three regimes the dynamics of goods market share \(\theta_c\) and the product market size \(L\) can be represented by a two-dimensional phase diagrams, since Lemma 1 shows that their dynamics do not depend on the market valuation share \(m_c\). In Figure 1, the three regimes are respectively represented by the three panels of the figure.

Characterizing the dynamics using the phase diagrams, we first notice that non-negativity of \(s_c\) and \(s_d\) in the simultaneous regime requires the following two (weak) inequalities, respectively

\[
\left(1 - \frac{1 - L}{\alpha L}\right)\kappa_c \leq \theta_c \leq \left(1 + \frac{1 - L}{\alpha L} \frac{\mu_c}{\mu_d}\right)\kappa_c.
\] (33)

We depict both “boundaries” of the simultaneous regime in Figure 1, panel b). Second, we notice that in the simultaneous regime both \(L\) and \(\theta_c\) are only feasible in the interval \((0, 1)\). Our case of interest arises if the zero-motion-loci of the phase diagrams fall inside the unit square, which requires the following assumption:\(^6\)

**Assumption 1.** \(\rho \left(\mu_c^{-1} + \mu_d^{-1}\right) < \alpha.\)

Under this assumption, there is a unique path towards a constant value of \(L\) in both the dirty-only regime and the clean-only regime. We indicate these paths by the red lines with arrows.\(^7\) In the simultaneous regime, there is an interior steady state, but it is unstable.

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\(^6\)In the alternative situation, with Assumption 1 violated, an equilibrium without innovation arises. For growth in steady state with innovation active in only one of the two sectors, it is sufficient that \(\rho < \alpha\mu_k\) for the innovation active sector \(k\). The more restrictive condition in Assumption 1 warrants a steady state with innovation active in both sectors.

\(^7\)Any path starting above the saddlepaths implies that \(L = 1\) in finite time, beyond which the arbitrage condition - i.e. the equation for \(\dot{L}\) in Lemma 1 - can no longer be satisfied; any path starting below the path implies that \(L\) vanishes which is inconsistent with the TVC.
We define a steady state (an asymptotic steady state) as an equilibrium in which $\theta_c, m_c$ and $L$ are constant (asymptotically constant). The phase diagrams allow us to immediately conclude the following:

**Proposition 1.** If $\sigma > 1$ and Assumption 1 hold,

1. the model has three steady states:
   
   (a) an interior steady state with simultaneous R&D in both sectors, characterized by $m^*_c = \theta^*_c = \kappa_c$, and $L^* = (1 + \alpha)^{-1}(1 + \frac{\rho}{\mu_d} + \frac{\rho}{\mu_c})$;
   
   (b) an asymptotic steady state with innovation in the dirty sector only: $m_c \to m^*_c = 0$, $\theta_c \to \theta^*_c = 0$, $L \to L^* = (1 + \alpha)^{-1}(1 + \frac{\rho}{\mu_d})$;
   
   (c) an asymptotic steady state with innovation in the clean sector only: $m_c \to m^*_c = 1$, $\theta_c \to \theta^*_c = 1$, $L \to L^* = (1 + \alpha)^{-1}(1 + \frac{\rho}{\mu_c})$;

2. the interior steady state is unstable, while the two corner steady states exhibit saddle path stability.

The two stable steady states can arise, because both consumers and producers benefit from market size. With high substitution between the two goods, clean and dirty ($\sigma > 1$), consumers relatively easily substitute towards the good that becomes relatively cheap. If the dirty sector starts with a relatively low price, consumers mainly spend their income on dirty goods, market size is big for these goods, and innovators realize a higher return on innovation in this market, thus lowering prices and reinforcing the incentive for consumers to mainly spend on these goods. The spending share of the sector keeps increasing so that ultimately spending on dirty goods completely dominates the market. This explains why the dirty-only steady state arise. However, since the two sectors are symmetric, the same intuitive reasoning can be provided for the clean good: a large market size for clean triggers innovation for clean and reinforces the clean market dominance – the clean-only steady state arises.
3.2 Overlap and Global Multiplicity

The existence of two saddle point stable corner steady states together with an unstable interior steady state is associated with self-fulfilling expectations and path dependency in the literature, see e.g. Krugman (1991) and Matsuyama (1991). We show that, similar to this literature, in our model both history and expectations can play a role in the selection of the long-run equilibrium. We find that for a large enough value of $\sigma$ there is a range of initial conditions ("overlap") for which self-fulfilling prophecies are possible. We start with the following definition.

**Definition 1 (Equilibrium Path).** An equilibrium path is a sequence of $(\theta_{c,t}, m_{ct}, L_t, s_{ct}, s_{dt})$ that satisfies (30), (31), or (32) at any point in time, and approaches one of the two saddle-point stable steady states as $t \to \infty$.

Since (30), (31), and (32) summarize the dynamic optimization of the firms and the households as well as the market clearing conditions, an equilibrium path is a sequence of $(\theta_{c,t}, m_{ct}, L_t, s_{ct}, s_{dt})$ that jointly maximizes firm profits and household lifetime utility, while clearing the factor, goods, and capital markets at the same time. Put in the context of rational expectations, an equilibrium path is the outcome of coordinated beliefs among all agents about all future actions of other agents and the resulting state of the economy. As agents are symmetric and atomistic, in equilibrium all agents must share the same belief, which can be represented by the equilibrium sequence of $(\theta_{c,t}, m_{ct}, L_t, s_{ct}, s_{dt})$. When agents coordinate on such a belief, it will turn out to be consistent with the optimizing behavior of all agents and market clearing of all markets at any point in time and thus consistent with agents’ belief. Multiplicity arises if for the same initial condition multiple such equilibrium paths exist. In this case, the economy is free to select any such path, meaning that agents can coordinate on any such beliefs that will turn out to be rational. In our deterministic setting of the model, once a selection is made at time 0, there is no more uncertainty and rational expectations collapse into perfect foresight.

We further refer to any equilibrium path that approaches the clean steady state a clean path, and one that approaches the dirty steady state a dirty path. Global multiplicity exists if from the same initial conditions, both a clean and a dirty path can be found so that in the long run, both the clean and the dirty steady states are consistent with rational expectations.

We characterize the global multiplicity in our model by finding, first, the full range of initial conditions (that is, the value of the pre-determined variable $\theta_c$), from which the clean steady state can be reached, and, second, the full range of initial conditions from which the dirty steady state can be reached. Our method is to pick any of the steady states and follow the associated dynamics back in time to trace out the initial conditions. We then show under which conditions there is an overlap between both ranges. From the initial conditions that belong to this overlap, both steady states can be reached, so that multiplicity and self-fulfilling expectations arise as in Krugman (1991).
For concreteness, consider the dirty-only steady state, which relates to (b) in Proposition 1 and is represented by the left corner in panel (a) of Figure 1. Conditional on staying in the dirty-only regime, this steady state is saddle-point stable and there is a unique path leading to it. As shown by Lemma 1, the equilibrium values along this path for $\theta_c$ and $L$ can be determined independently from $m_c$. Since close to the corner steady states $m_c$ is far from the regime switching threshold $\kappa_c$ and it takes time for $m_c$ to adjust, the saddlepath indicated by the red arrows in panel (a) of Figure 1 already traces at least a section of the initial conditions ($\theta_c$), from which the steady state can be reached.

However, away from the steady state the value of $m_c$ will be different. Sufficiently far away from the dirty-only steady state the dirty-only regime might not be an equilibrium. We can only trace the global saddlepath by taking into account the dynamics of all three variables, $\theta_c$, $L$, and $m_c$, since the saddlepath associated with the steady state has to be characterized in three-dimensional space. We trace the three-dimensional saddlepath by “starting” arbitrarily close to a steady state and going back in time, using the equations of motion in Lemma 1, to identify the preceding values of $\theta_c$, $L$, and $m_c$. We continue until we find the point in time for which $m_c$ reaches the value $\kappa_c$, at which time there must be a regime switch. As a regime switch is also associated with the switch in direction of change for $\theta_c$, we have thus traced all initial conditions that can lead to the steady state.

Figure 2 illustrates the $\dot{m}_c = 0$ surface in the three dimensional space. The surface has a flat part at $m_c = \kappa_c$ that represents the admissible area of the simultaneous regime defined by (33). Above (below) the plane, $m_c$ increases (decreases), and the forward-looking market share of the clean sector increases (decreases) relative to that of the dirty sector.

![Figure 2: The $\dot{m}_c = 0$ Surface](image)

Below the $\dot{m}_c = 0$ surface in the unit cube, there is a unique path that leads to the dirty-only steady state. Above it there is a unique path that leads to the clean-only steady state. The resulting pair of equilibrium paths for a numerical example provided in Section 4
is shown in Figures 6, where each panel represents a different set of parameter values. The two paths may share values of $\theta_c$, such that from these shared values both the equilibrium path to the clean steady state and the equilibrium path to the dirty steady state can be reached. In line with Krugman (1991) we label this range the “overlap”.

The overlap becomes easier to inspect when we project the two equilibrium paths onto the $(\theta_c, m_c)$-plane, as in Figure 3. Along the equilibrium path in the dirty-only regime, market share $\theta_c$ ranges from zero to the maximum value that we identified by the reverted-time procedure outlined above and that we denote $\theta_c^{DS}$. Similarly, in the clean-only regime the market share ranges from the identified minimum value, denoted $\theta_c^{CS}$, to 1. The “overlap” is the range $[\theta_c^{CS}, \theta_c^{DS}]$, the non-emptiness of which we still need to prove, such that for any initial value of $\theta_c \in [\theta_c^{CS}, \theta_c^{DS}]$ both steady states can be reached. The selection of the steady state constitutes a coordination problem which is solved outside the model, such as by means of a sunspot. Expectations here refer to entrepreneurs’ belief about all future relative, equilibrium market size of the two sectors, which serves to pin down the forward-looking market valuation share $m_c$ at time zero, that is, at the time of the initial condition. Given the deterministic nature of the model and the saddle point stability of the corner steady states, for each of the corner steady states, the relative equilibrium market size at any preceding point in time is deterministic and can be traced backward. Expectations, therefore, are in essence the belief about which steady state is to be reached in the long run. Expectations are self-fulfilling as long as we start in the overlap: if everybody expects the clean steady state to be reached, it will be reached since it is consistent with the equilibrium conditions of the model.

![Figure 3: Two-Dimensional Projection](image-url)

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8 This three-dimensional phase diagram is the counterpart of the two-dimensional spiral figure in Krugman (1991).

9 Figure 4 presents the corresponding projections for the numerical examples.

10 Starting from the corner steady states and tracing equilibrium dynamics backward in time, $\theta_c^{DS}$ and $\theta_c^{CS}$ are given by the value that $\theta_c$ takes when $m_c = \kappa_c$ in the dirty-only and clean-only equilibrium, respectively.
The overlap does only arise if consumers see clean and dirty goods as sufficiently strong substitutes as we state in the following proposition:

**Proposition 2.**

1. *If and only if* $\sigma > 2$, *the equilibrium paths towards the two corner steady states overlap for a range of initial states* $\theta_c$.

2. *The overlap increases with* $\sigma$.

If the initial market share for clean goods, as measured by $\theta_c(0)$, is relatively small, innovation in the clean sector pays off little in the *near future*. An equilibrium with only innovation in the dirty sector will thus result. However, if innovators expect that the *far future* clean-sector market size will be relatively large, clean-sector innovation may pay off more than dirty-sector innovation. This creates a self-fulfilling prophecy: if all innovators expect high returns, they invest and the future market size will be big, thus justifying the expectations.

Self-fulfilling prophecies only arise with sufficiently good substitution. In case of poor substitution, innovation in one sector substantially decreases the relative price in this sector and reduces the return to innovation. Simultaneously, innovation in the sector increases productivity in that sector and thus tends to increase the return on investment. Whether innovation is profitable or not depends on which of the two effects dominates. Only if substitution is not too poor and productivity is sufficiently responsive to technical change, the return to innovation in the initially small sector can remain relatively high as compared to innovation in the bigger sector so that both types of innovation can arise in equilibrium. In our model with final good being linear to average sectoral quality levels, the critical value for the substitution elasticity $\sigma$ turns out to be 2. A more general case is considered in Appendix II.

If consumers consider clean and dirty goods more substitutable, a small change in relative prices causes a big shift towards the cheaper good. Hence, a switch in innovation from dirty to clean (or the other way around) has more immediate effects in the short run if substitution is better. Loosely speaking, the short run gains kick in quicker. Thus, if initial market share for clean is initially small, the returns to clean innovation take too much time to become rewarding for small $\sigma$ but become quickly rewarding for high $\sigma$. This explains why the scope for self-fulfilling prophecies (as measured by the width of the overlap) increases with substitution possibilities.

### 3.3 Fast versus Delayed Transition

So far we have discussed under which conditions the clean-only regime and the dirty-only regime both constitute an equilibrium. In this subsection we show that there is a further
multiplicity: if for a given initial condition an equilibrium path, along which clean innovation is exclusively active the entire time, co-exists with one where dirty innovation is exclusively active along the entire path, then there is (at least) one further equilibrium path in which the two types of innovation are sequentially or even simultaneously profitable. When firms invest in both types of R&D over time, transition to the steady state is slower than when only one type of R&D occurs in equilibrium, since with innovation in both sectors the productivity differences across the sectors remain more balanced, while ultimately only one sector can serve the entire market (asymptotically).

We now analyze under which initial conditions, in terms of $\theta_c(0)$, the simultaneous regime can be an equilibrium and how these initial conditions overlap with the conditions derived in the previous subsection.

We first note that simultaneous R&D can never last long as one sector must start to dominate whenever $\sigma > 1$. More formally, this is because the simultaneous equilibrium is instable. We can derive the following about when the economy can be in the simultaneous regime:

**Lemma 2.** (i) If the economy is in the simultaneous regime at time $t$ and $\theta_c(t) \neq \kappa_c$, it must leave this regime in finite time after $t$. (ii) If the economy is not in the simultaneous regime at time $t$ and no unexpected shocks occur, it cannot be in the simultaneous regime after $t$.

We next note that if $\sigma < 2$, we have no overlap, and for initial conditions close to the interior steady state, there must be simultaneous innovation. This is intuitive: with relative market size (as measured by $\theta_c$) roughly proportion to relative R&D cost (as measured by $\kappa_c$), both types of R&D should be viable for a while. That is, if all innovation effort would be directed towards expanding only a single sector, consumers would not be willing to shift so much to this sector and instead innovation is spread over the two sectors. The same logic could apply for a bigger value for $\sigma$, as long as total labor available for innovation would be big. We can show that indeed simultaneous R&D is an equilibrium even for $\sigma > 2$.

**Proposition 3.** With symmetry ($\mu_d = \mu_c$) and starting from an initial $\theta_{c,0} \in \theta^{CS}_c, \theta^{DS}_c$, the transition towards one of the corner steady states could follow multiple possible trajectories.

1. A fast transition, in which the economy selects immediately the corner stable path, is always possible.

2. The number of different patterns of delayed transition depends on the value of $\sigma$, relative to two critical values, $\bar{\sigma}$ and $\tilde{\sigma}$, where $2 < \bar{\sigma} < \tilde{\sigma}$. Before finally following the corner stable path,

(a) if $2 < \sigma < \bar{\sigma}$, temporary simultaneous R&D is the only possible delay;
(b) if $\sigma \geq \bar{\sigma}$, delay possibilities include temporary simultaneous R&D and temporary regime switches between the two corner innovation regimes;

(c) if $\sigma > \bar{\sigma}$, delay must include temporary stagnation with no R&D.

The above proposition makes clear that not only the long run equilibrium itself but also the transition towards it requires a coordination of beliefs. If collectively entrepreneurs in the economy believe that a bright green future is just around the corner, such optimism can spur clean innovation such that the economy is immediately on the fast transition path towards the clean steady state. If entrepreneurs believe that there will be some period of indecisiveness concerning the relative strength of the two technologies and their profitability, despite one of them being the only viable technology in the long run, the economy invests accordingly to reap any short and mid term gains that can still be harvested by investing into both technologies, before switching definitively to the only viable future technology. Similarly, if clean technology is believed to be the only viable technology in the long run with dirty technology still profitable in the short and mid run, entrepreneurs will still invest into dirty technology before everyone switches to clean at the same time. And finally, if it is believed that investing into clean technology is the only profitable strategy while such profitability will only be manifested starting from some future point of time on, no rational investor will make any investment in either of the two technologies until the time is ripe to invest into clean technology. Upon arrival of such time, everyone starts to invest into the only future viable technology, fulfilling the initial expectations.

The possibility of delayed transition depends on the value of $\sigma$ which reflects the role of market responsiveness. For a relatively low value of $\sigma$, relative demand responds to the innovation induced changes in relative price only sluggishly. No matter which sector is expected to eventually dominate, changes towards this sector can occur only slowly. It takes a lot of cumulative changes to finally tip the balance towards one of the two sectors decisively. As $\sigma$ becomes higher, the speed of demand responses picks up. An initial bias of expectations in favor of one the two sectors can now be much more easily strengthened and amplified by market responses, and become self-reinforcing. This higher speed of demand response, however, also adds to volatility, as it becomes easier to tip the balance in favor of a specific sector. For a very high value of $\sigma$, relative demand now responds to changes in relative price vigorously. A small initial bias of expectations can be easily strengthened and amplified by market responses, and become self-reinforcing.

4 Numerical Example

We have so far shown analytically that within the overlap of initial conditions, both the long run equilibrium and the transition towards it are indeterminate. Two economies that start from the same market share of their green sector may converge to different steady states, with one economy innovating only in clean and the other one only in dirty technology,
with nothing but entrepreneurs’ collective sentiments and expectations being the ultimate path selection device. Differences in expectations also pin down how fast the economy is converging to the selected steady state, giving rise to rich patterns of self-fulfilling beliefs. In this subsection, we provide a simple numerical example to illustrate the overlap and transition patterns, and how they changes with the parameter $\sigma$.

We set the factor share of labor to the commonly used value of two-third so that $\alpha = \frac{1}{3}$. We then calibrate the long run growth rate of the model, $g = \mu_k(1 - L_k) = \frac{\alpha \mu_k - \rho}{1 + \alpha}$, to 2 percent using $\rho$ and $\mu_k$, where $k$ refers to the innovation active sector. Given $\rho$, the required research labor efficiency parameter $\mu_k$ can be calculated as $\mu_k = \frac{(1 + \alpha)g + \rho}{\alpha}$. To avoid the stagnant case of no growth, $L^* < 1$ must hold in all three steady states, which requires $\rho < g(1 + \alpha)\mu_d/\mu_c$. With $g = 0.02$, we thus set $\rho = 0.01$, and calibrate the baseline research labor efficiency parameters to be $\mu_c = \mu_d = 0.11$. For the last parameter of the model, the elasticity of substitution, empirical estimates tend to vary by a large range. In general, estimates based on production input substitution tend to offer smaller numbers, compared to estimates of product substitution. For the elasticity of substitution between clean and dirty inputs at the aggregate level, Papageorgiou et al. (2016) suggest values around 2 in the energy-generating sector and values close to 3 in nonenergy industries. For product substitution, the estimates by Hottman et al. (2016) range from 4.7 to 17.6 with a median elasticity of 6.9. Here for the illustration of the overlap, we allow $\sigma$ to vary between 1.5 and 7, but use larger values (up to 14) to illustrate different transition trajectories.

Starting from each of the two corner steady states, we simulate the saddlepath by going backward in time until the variable $m_c$ approaches the regime-separating value of $\kappa_c$. The result is provided in Figure 4, where the red and blue curves are the trajectories starting from the clean and dirty steady state, respectively. For all $\sigma$ values larger than 2, there are a region of $\theta_c$, from which both saddlepaths can be reached. This region grows larger with increasing $\sigma$, as suggested by Proposition 2. Figure 5 in addition shows the changes of the overlap with varying values of $\mu_c$, while holding $\mu_d$ constant at 1 and $\sigma$ constant at 5. As $\mu_c$ increases, the overlap shifts towards the left, meaning that as clean research becomes more productive, the clean path can be reached from a more disadvantageous initial position for the clean sector. Further, it can also be seen from Figure 5 that the overlap becomes smaller the higher the difference in research productivities between the two sectors. Figure 6 in addition gives the equilibrium paths for different $\sigma$ values. And finally, Figure 7 provides the slow transition paths with regime switches and stagnation for high $\sigma$ values.
Figure 4: Overlap with Different $\sigma$ Values

Figure 5: Overlap with Varying $\mu_c$ Values
Figure 6: Transition Paths
Figure 7: Slow Transition with Regime Switch and Stagnation
5 Policy Implications

The existence of overlap suggests that even with unfavorable initial conditions, supportive expectations and optimism alone could help select the clean innovation path leading to the desirable long-run equilibrium of green growth. From the policy perspective, however, it is the cases when expectations are pessimistic or oscillating that we need to be concerned about. While policy measures that stimulate optimism towards clean innovation might sound speculative, we may also reframe the policy question as how to eliminate the undesirable outcome of a self-fulfilling dirty path. In such case, not only policy intervention will be needed, but the existence of overlap further implies that any environmental policy also needs to be sustained sufficiently long in order to eliminate any future reverse of the innovation trend such as due to an expectation shock. Unless environmental policy is sustained long enough to push the economy outside of the overlap, even if policy interventions could help push the clean technology frontier so much forward that the clean technology is leading, the economy might revert to the dirty path by pessimistic expectations. The often discussed self-fulfilling debt crises are classic examples of such risks. Together with any policy interventions, therefore, there needs to be a minimum commitment period for the policy to eliminate future expectation risks.

In this section, we discuss how different policy measures could affect the overlap and the equilibrium paths, and what is needed for such policy measures to eliminate the dirty innovation path. Before we proceed to discuss the different policy instruments, we introduce the following definitions:

**Definition 2 (Minimum Policy Stringency).** The minimum policy stringency is defined as the constant level of policy (tax rate, subsidy rate, governmental research employment, etc.), denoted by $\tau$, with which $\theta_{DS}(\tau) = \theta_{c,0}$,

$$\theta_{c,0}^{DS}(\tau) = \theta_{c,0}, \quad (34)$$

where $\theta_{c,0}^{DS}(\tau)$ denotes the upper bound of the overlap under the constant $\tau$.

**Definition 3 (Minimum Policy Commitment Period).** The minimum commitment period $T$ is defined as the period of time necessary under the minimum policy stringency to fully eliminate future expectation risks. That is,

$$\theta_{c,0} + \int_{0}^{T} \theta_{c,t}(\tau) \, dt = \theta_{c}^{DS}, \quad (35)$$

where $\theta_{c}^{DS}$ denotes the upper bound of the overlap in laissez faire.

Together, the above definitions describe what is needed for a policy to eliminate the dirty innovation at present time as well as makes future expectation reversion impossible. We now proceed to different policy instruments. We start with the often discussed research subsidy (tax) to the clean (dirty) innovations, and then move on to government-funded clean research and infrastructure investment as alternative policies.
5.1 Dirty Research Tax or Clean Research Subsidy

One possible policy intervention to eliminate the dirty equilibrium path is simply changing the relative costs of doing research in the two sectors by introducing a dirty research tax or a clean research subsidy.\footnote{Subsidies can be financed either through a lump-sum tax or a proportional labor income tax on the households. In either case, raising governmental revenue does not distort households’ decisions.} For both tax and subsidy, a minimum commitment period of the policy is needed in order to fully eliminate the risk of reverting to the dirty path due to adverse expectation. To eliminate such expectation risk, the policy needs to be sustained until the economy moves out of the overlap region. Comparing tax with subsidy, while both options are theoretically equivalent, a dirty research tax would appear more appealing in terms of implementation, as rational agents will never coordinate on a dirty path, foreseeing the tax payments and the resulting high innovation costs. However, a dirty research tax has a few practical drawbacks. First, if the clean research labor is relatively unproductive and the clean sector is sufficiently lagged behind, i.e. if $\mu_c$ and $\theta_c$ are both low, research labor in the dirty sector will not be directed into research in the clean sector by the dirty research tax, but rather to production. The reason is that when research labor in the clean sector is too unproductive, the low growth potential in future consumption by innovation does not justify the sacrifice in current production and consumption. The effect will exacerbate if the households are more impatient (higher $\rho$). Blocking dirty research without providing additional help to clean research will thus be very costly. Second, as a dirty research tax does not create any distortion should a clean path be chosen, this policy does not generate higher innovation for the clean path as compared to laissez faire. A clean research subsidy, on the contrary, effectively increases the research labor productively in the clean sector relative to production, and thus also directs labor from production to clean research. As a result, the growth rate under a clean research subsidy will be higher compared to with dirty research tax, and one can thus expect a faster transition and a shorter minimum commitment period with the subsidy.

The above comparison is confirmed by the numerical example, where for simplicity, have assumed a constant tax ($\tau_d$) or subsidy rate ($\tau_c$) on the costs of hiring research labor in the dirty and clean sector, respectively. With a constant $\tau_d$ or $\tau_c$, $\kappa_c$ in the regime-separating condition changes to

$$\kappa_c = \frac{1}{1 + \mu_c \zeta_{tx}}, \quad (\zeta_{tx} = 1 + \tau_d > 1) \tag{36}$$

$$\kappa_c = \frac{1}{1 + \mu_c \zeta_{sb}}, \quad (\zeta_{sb} = \frac{1}{1 - \tau_c} > 1). \tag{37}$$

As long as a tax or subsidy is in place, the regime border shifts downwards along the $\theta_c$ scale and intersects with the dirty path at a lower $\theta_c$ value. This implies that with a dirty research tax or a clean research subsidy, the overlap is shift down towards lower value of $\theta_c$, and the dirty path can only be reached from a smaller range of initial conditions.
Using the same numerical example as before, Table 1 shows the minimum policy stringency and commitment period needed for the dirty research tax and clean research subsidy. Examining the numbers, we see a few clear patterns. First, for very low $\theta_c$, imposing high dirty research tax only creates a (near) zero-growth trap. This problem, however, does not arise under a clean research subsidy, as now the research labor productivity $\mu_c$ is augmented by the subsidy. Second, for the same initial conditions that allow positive growth rate under both tax and subsidy, the required minimum policy stringency for both tax and subsidy could be substantial, but does decline rather fast with increasing initial advance of the clean sector ($\theta_{c,0}$). Third, when both tax and subsidy can be used to eliminate the dirty innovation path, the minimum commitment period is much longer for a dirty research tax than a clean research subsidy, as expected. And finally, as $\sigma$ becomes larger and the clean and dirty goods become increasingly substitutable, for any initial $\theta_{c,0}$, the required minimum policy stringency increases. This is because with increasing substitutability of the two sectors, the market evaluates any changes in relative technological advancement more cautiously ($m_c$ changes more slowly compared to $\theta_c$), and the persistence in a given evaluation is stronger. To eliminate a pessimistic market evaluation, therefore, more stringent policy is needed. The good news is, on the other hand, when the two final goods are more substitutable, the relative market size $\theta_c$ does change faster, which should shorten the minimum commitment period. This latter effect can also be observed in the numbers provided with the exception of at high initial $\theta_c$. There, the minimum commitment period first increases and then drops, as $\sigma$ increases. This is because as $\sigma$ gets larger, the size of the overlap also increases, which tends to increase the minimum commitment period. This latter effect is, however, only relevant for relatively large initial $\theta_{c,0}$.

Figures 8 and 9 provide in addition a graphic illustration of how the dirty innovation paths can be eliminated by a dirty research tax and a clean research subsidy, respectively. In the two figures, the magenta colored lines are the clean path with policy (tax or subsidy) in place, the red lines the clean paths after policy is abolished, and the blue lines the dirty paths. Again, it is visible from these figures that for very low $\theta_{c,0}$, the green growth path cannot be reached by imposing high dirty research tax. When both tax and subsidy can be used, the subsidy scenario features more drastic changes (higher innovation compared to laissez faire) along the transition.
Initial $\theta_c = 0.1$

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**Tax**

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<td>$\sigma = 7$</td>
<td>n.g.</td>
<td>n.g.</td>
<td>n.g.</td>
<td>949%</td>
<td>32.9</td>
<td>580%</td>
<td>19.1</td>
</tr>
<tr>
<td>$\sigma = 4$</td>
<td>95%</td>
<td>14.8</td>
<td>89%</td>
<td>14.5</td>
<td>82%</td>
<td>14.0</td>
<td>71%</td>
</tr>
<tr>
<td>$\sigma = 5$</td>
<td>96%</td>
<td>11.0</td>
<td>92%</td>
<td>10.8</td>
<td>86%</td>
<td>10.4</td>
<td>78%</td>
</tr>
<tr>
<td>$\sigma = 7$</td>
<td>97%</td>
<td>7.3</td>
<td>94%</td>
<td>7.2</td>
<td>90%</td>
<td>6.9</td>
<td>85%</td>
</tr>
</tbody>
</table>

Other parameters: $\alpha = \frac{1}{3}$, $\rho = 0.01$, $\mu_c = \mu_d = 0.11$ (n.g. = no growth)

Table 1: Minimum Policy Stringency and Commitment Period with Dirty Research Tax or Clean Research Subsidy
Figure 8: Eliminating the Dirty Path with Dirty Research Tax ($\mu_r = 1$)

Figure 9: Eliminating the Dirty Path with Clean Research Subsidy ($\mu_r = 1$)
5.2 Government-funded Clean Research

Instead of affecting private innovation decisions monetarily, policy makers may also make use of the spillover effects of research and fund clean research directly. Let $s_g$ denote the amount of research labor hired by the government for conducting clean innovation. For simplicity, we consider a constant $s_g$. The wage for the government funded research labor is given by $w_s g$, and can again be financed through a lump sum tax or a labor income tax from the household. With $s_g > 0$, the dirty research labor and the production labor are now bounded by $s_d \leq 1 - L - s_g$ and $L \leq 1 - s_g$, respectively. Thus, depending on the size of $s_g$, governmental research not only squeeze labor out of dirty research, but could also crowd out production labor. In other words, governmental clean research could have both composition and growth effects. With this two-fold effect in mind, we now distinguish the following innovation regimes:

$$
\tilde{Q}_r = \begin{cases} 
\mu_c (1 - L) & \text{if } 0 \leq m_c \leq 1, L = 1 - s_g (s_c = s_d = 0) \\
\mu_c (1 - L) & \text{if } m_c > \kappa_c, L < 1 - s_g (s_c > 0, s_d = 0) \\
\alpha \mu_d L [(1 + \mu_r)\theta_c - 1] & \text{if } m_c = \kappa_c, L < 1 - s_g (s_c, s_d > 0) \\
-\mu_d (1 - L) + (\mu_c + \mu_d) s_g & \text{if } m_c < \kappa_c, L < 1 - s_g (s_c = 0, s_d > 0)
\end{cases}
$$

(38)

where in the above equations $s_c$ represents the private clean research effort.

The existence of governmental clean research substantially complicates the dynamics within the dirty research regime ($s_c = 0, s_d > 0$), as the equilibrium path might no longer be monotone. There are potentially many ways the dirty equilibrium path can still be reached, despite the temporary governmental research. A permanently sustained governmental clean research can however eliminate the dirty innovation path altogether.

In the following, Figure 10 shows the minimum policy stringency required for different initial $\theta_{c,0}$ at $\sigma = 4$ and $\sigma = 5$, and the share of governmental research in total clean research within the minimum commitment period. (Each period in this figure corresponds to one month.) The green line is the private clean research effort, the red line the minimum policy stringency $s_g$, the dashed magenta lines is the corresponding share of governmental research in total clean research, and the black line is the total production labor. For very low initial $\theta_c$, the figure shows that the minimum policy stringency requires an clean innovation effort above the level that is privately profitable. Under such circumstances, there will be no private innovation, and all clean research is done by the government. Over time, as $\theta_c$ increases along the clean path, the share of governmental research in total clean research declines. Once this share falls below 1, governmental clean research does not spur more clean research in total, but simply replaces part of the private research effort. The role of governmental research, in such cases, lies rather in deterring undesired dirty innovation.
The effect of eliminating the dirty innovation path is illustrated by Figure 11. The interpretation of the figure is similar to that in last section with a dirty research tax or clean research subsidy. The magenta part of the clean path illustrates the minimum commitment period.
5.3 Infrastructure Investment

A final policy measure we consider in this section is governmentally funded infrastructure investment. Suppose now consumption combines the private consumption good with infrastructure so that the composite consumption good is given now by

\[
C = \left[ (I_c C_c)^{\frac{\sigma-1}{\sigma}} + (I_d C_d)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}},
\]

where \( I_c \) and \( I_d \) are the infrastructure in sector \( c \) and \( d \). A few common examples of such infrastructure that is combined with private goods in consumption include roads, ICT network and concert halls. Specifically in the context of clean and dirty consumption, we can think of charging station infrastructure of electric cars vs. conventional gas stations, bike lanes vs. parking space, etc.

Suppose further that the infrastructure is typically maintained by the government, such that

\[
\dot{I}_j = \gamma_j i_j I_j \quad j \in \{c, d\},
\]

where \( i_j \) is the labor hired by the government for building and improving the infrastructure of sector \( j \), and \( \gamma_j \) is the respective productivity parameter of the workers in the two sectors. The total available labor is now to be divided among the production of final goods, building.
infrastructure, and researching in private good production technology of the two sectors. That is,

\[ 1 = L + s_c + s_d + i_c + i_d. \]  

(41)

In addition, compared to the baseline, the state variable \( \theta_c \) and its change over time are now given by

\[
\theta_c = \frac{L_r}{1 + L_r} = \frac{(I_r Q_r)^{\sigma - 1}}{1 + (I_r Q_r)^{\sigma - 1}} \tag{42}
\]

\[
\dot{\theta}_c = \theta_c (1 - \theta_c) (\sigma - 1) \left( \hat{I}_r + \hat{Q}_r \right), \tag{43}
\]

where \( I_r = \frac{L_r}{I_d} \) and \( \hat{I}_r = \gamma_c i_c - \gamma_d i_d \) give the relative infrastructure strength of the two sectors and its growth rate.

With centrally managed infrastructure, an apparent measure for eliminating the dirty innovation path would be to invest into the clean infrastructure only \((i_c > 0, i_d = 0)\). This policy is qualitatively similar to the case with governmentally funded clean research, as can be seen by a simple mental exercise of setting \( \gamma_c = \mu_c \). All qualitatively results from last section should thus also hold in the case of governmentally funded clean infrastructure investment.

6 Conclusions

In this paper we investigate the role of expectation in directed technical change towards green growth. Using a standard workhorse model, we show that with sectoral spillover of innovation and forward looking investment decisions, the market size effect is no longer the only deciding factor in determining the future innovation direction in laissez faire. Rather, there exists a considerable range of initial conditions (the so called “overlap”), which allow multiple stable equilibrium paths towards opposite corner steady states. The existence of overlap highlights the role of expectation as a coordination device, and the importance of optimism in shaping the green growth future. Nonetheless, as we have often witnessed from the financial market booms and busts, the coordination of expectation could also impose considerable risks in reverting sentiments and development trend. This feature of expectation led coordination of innovation behavior is highly relevant for environmental policy making. By discussing a few common policy instruments, we show in this paper how and what is required to eliminate such expectation risks in directing future innovation towards clean innovations.

The present work also serves as a starting point for various future extensions. We have investigated the role of expectation in a deterministic model, with the only uncertainty concerning the choice of belief at time zero. It is a natural extension to consider how the overlap changes if agents form stochastic expectations instead. Relatedly, extending the model into a global game setting and investigating the coordination outcomes where agents
have incomplete information will surely yield further interesting insights. Further points of interest for future work include considerations concerning the interplay between heterogeneity (such as financial constraints) and expectations, and demand side complementarity such as due to consumers’ other regarding preferences.
References


Appendices

A Deriving supply side equations

Substituting the firm’s demand curve (2) and research technology (3), we write the Hamiltonian for the firm’s maximization problem as:

\[ H_{ji} = \alpha P_j L_j^{1-\alpha} q_{ji} x_{ji}^\alpha - P_j q_{ji} x_{ji} - w s_{ji} + \lambda_{ji} \mu_j Q_j s_{ji}. \]

The first-order condition for \( s_{ji} \) is (6). The first-order condition for \( x_{ji} \) reads

\[ \alpha P_j L_j^{1-\alpha} q_{ji} x_{ji}^\alpha = P_j q_{ji}, \]

which after substitution of (2) gives (5) and which after cancelling the \( P_j q_{ji} \) term at both sides gives (12). Substituting these results into the definition of profits gives

\[ \pi_{ji} = (1/\alpha - 1) P_j x_{ji} q_{ji} \]

so that, by the envelope theorem, \( \partial \pi_{ji} / \partial q_{ji} = (1/\alpha - 1) P_j x_{ji} \). This gives the optimality condition (7). Since both \( \partial \pi_{ji} / \partial q_{ji} \) and \( x_{ji} \) are the same across firms, we can write (11). We find (8) from substitution of (12) into the production function for \( Y_j \). We find (9) from sustitution of (8) into (1).

To derive (14), we start from (7) and omit the subscript \( i \) in the light of (11), divide by \( \lambda_j \) and substitute (12) to arrive at

\[ \frac{\dot{\lambda}_j}{\lambda_j} = \frac{1}{\alpha} = \frac{\mu_k}{P_k} \]

where \( \alpha = (1/\alpha)^{\frac{1}{1-\alpha}} \). Recalling that sector \( k \) is the research active sector, we can write from (6) and (9)

\[ 1 = \frac{\lambda_k \alpha \mu_k}{\alpha \lambda P_k}. \]

Multiplying the last term in (A.1) by the ratio in (A.2), we arrive at:

\[ \frac{\dot{\lambda}_j}{\lambda_j} = \frac{1}{\alpha} = \frac{\mu_k L_j P_k \lambda_k}{P_k \lambda_j}. \]

Using (9) to eliminate \( P_j / P_k \), we find:

\[ \frac{\dot{\lambda}_j}{\lambda_j} = \frac{1}{\alpha} = \frac{\mu_k L_j Q_k \lambda_k}{Q_j \lambda_j}. \]

From (A.4) we find (14) by setting \( j = k \) and replacing the left-hand side by \( \dot{w} - \dot{Q}_k \) from (6)
B Proof Lemma 1

From (A.4), using (22) to eliminate $L_j$, we find the change in the relative shadow prices:

$$\hat{\lambda}_k - \hat{\lambda}_j = \mu_k \alpha L \left[ \theta_j \frac{Q_k \lambda_k}{Q_j \lambda_j} - \theta_k \right]. \quad (B.5)$$

For the clean-only and dirty-only regimes, we derive the equation of motion for $m_c$ by using (B.5) and (19) to eliminate the shadow prices and innovation rates, respectively, in (29). The equation of motion for $\theta_c$ follows from (19) and (23); the equation of motion for $L$ follows from (19) and (25).

In the simultaneous regime, (14) holds for both $k = c$ and for $k = d$ since both sectors are research active. Together with (29), this allows us to write:

$$\dot{m}_c = 0 \Leftrightarrow \hat{Q}_c - \hat{Q}_d = \hat{\lambda}_d - \hat{\lambda}_c = \alpha L (\mu_c \theta_c - \mu_d \theta_d). \quad (B.6)$$

Substituting $\hat{Q}_c = \mu_c s_c$ and $\hat{Q}_d = \mu_d (1 - L - s_c)$, both of which follow from (10) and (19), into (B.6) gives $s_c = (1 - \kappa_c)(1 - L) + (\theta_c - (1 - \kappa_c))\alpha L$, i.e. the solution for $s_c$ in terms of $\theta_c$ and $L$. Setting $k = c$ in (25) and using the derived $s_c$ expression to eliminate the innovation rate, we find the equation of motion for $L$. Substituting (B.6) into (23), we find the equation of motion for $\theta_c$.

C Proof Proposition 1

The derivation of the three steady states from (30)-(32) is straightforward. The steady states correspond to the intersection of the $\dot{\theta}_c = 0$ and the $\dot{L} = 0$ loci in the three phase diagrams in Figure 1. While from the phase diagrams it is also obvious that the stability claim in the second part of Proposition 1 holds within each regime, to take possible regime switch into account we log-linearize the model around the steady states. Log-linearization around the interior steady state gives

$$\begin{bmatrix}
\dot{\hat{\theta}}_c \\
\dot{\hat{L}} \\
\dot{\hat{m}}_c
\end{bmatrix} = \begin{bmatrix}
(\sigma - 1)\alpha \mu_d (1 - \theta^*_c) L^* & 0 & 0 \\
0 & (1 + \alpha) \kappa_c \mu_d L^* & 0 \\
0 & 0 & \alpha \mu_c \theta^*_c L^*
\end{bmatrix} \times \begin{bmatrix}
\hat{\theta}_c \\
\hat{L} \\
\hat{m}_c
\end{bmatrix}. \quad (C.7)
$$

Since obviously the three eigenvalues of $M_s$ are all positive, a small deviation from the interior steady state will take the system steadily away from it. The positive eigenvalue associated with $m_c$ again highlights the knife-edge characteristics of the simultaneous regime – any perturbation will take the economy away from the simultaneous regime.
The curves illustrated correspond to the red path in Figure 1 Panel a) with different $\sigma$’s.

Figure 12: Projection of the Dirty Equilibrium Path in $(\theta_c, L)$ Plane

For the clean-only steady state, log-linearizing (30) we get the following:

$$
\begin{bmatrix}
\dot{\theta}_c \\
\dot{L} \\
\dot{m}_c
\end{bmatrix}
= 
\begin{bmatrix}
-(\sigma-1)\mu_c(1-L^*c) & 0 & 0 \\
\alpha\mu_cL^*c & (1+\alpha)\mu_cL^*c & 0 \\
-\alpha\mu_cL^*c & 0 & [(1+\alpha)L^*c-1]\mu_c
\end{bmatrix}
\begin{bmatrix}
\tilde{\theta}_c \\
\tilde{L} \\
\tilde{m}_c
\end{bmatrix}.
$$

(C.8)

It is obvious that the diagonal elements of the matrix $M_c$ are the three eigenvalues, indicating the clean-only steady state is saddle point. There is one saddle path going through the clean-only steady state and along the path, the economy will stay in the clean-only regime and eventually converge to the clean-only steady state. Similarly, the dirty-only can also be shown to exhibit saddle point stability.

D Proof Proposition 2

An overlap exists if $\theta_c^{CS} < \theta_c^{DS}$. For the existence of the overlap it is sufficient to show that $\theta_c^{DS} > \kappa_c$ and $\theta_c^{CS} < \kappa_c$ if and only if $\sigma > 2$. For the size of the overlap, we show that $\theta_c^{DS}$ increases with $\sigma$, while $\theta_c^{CS}$ decreases with it. We prove the two claims concerning $\theta_c^{DS}$: (i) $\frac{\partial\theta_c^{DS}}{\partial\sigma} > 0$, and (ii) $\theta_c^{DS} > \kappa_c$ if and only if $\sigma > 2$. The claims for $\theta_c^{CS}$ follow from similar argument.

Lemma 3. Consider the projection of the dirty equilibrium path in the $(\theta_c, L)$ plane within the dirty regime. The equilibrium path with a higher $\sigma$ value has a lower projection in the $(\theta_c, L)$ plane than one with a lower $\sigma$, while passing through the same steady state $(\theta_c^{*d}, L^{*d})$, as shown in Figure 12.

Proof of Lemma 3. The steady state given by Proposition 1 does not change with $\sigma$. Thus the equilibrium paths corresponding to different $\sigma$ values all pass through the same steady
state. In the $\theta_c$-$L$ plane (see Figure 12), the slope of the projection of the dirty equilibrium path in the dirty-only regime satisfies

$$\frac{\dot{L}}{\theta_c} = \frac{L}{\sigma - 1} \left[ \frac{(1 + \rho/\mu_d) - (1 + \alpha - \alpha \theta_c) L}{\theta_c(1 - \theta_c)(1 - L)} \right] \equiv f^{L\theta_D}(\theta_c, L, m_c, \sigma). \quad (D.9)$$

Evaluated at any given $(\theta_c, L)$ pair, this slope decreases with $\sigma$ given that the term in square brackets is positive along the dirty-only equilibrium path. This implies that equilibrium paths with different values for $\sigma$ can cross only once, which is at the steady state. Hence the equilibrium path becomes flatter as $\sigma$ becomes higher.

The implication of Lemma 3 is that while considering the equilibrium paths of different $\sigma$, the path with a higher $\sigma$ has a lower $L$ for the same $\theta_c$, and a higher $\theta_c$ for the same $L$ within the dirty regime. We now proceed with the proof of the proposition.

**Proof of Proposition 2.** Consider the projection of the dirty equilibrium path in the $(\theta_c, m_c)$ plane within the dirty regime. From the phase diagram, we know $\dot{m}_c/\dot{\theta}_c > 0$. Further, the slope of the projection is given by

$$\frac{\dot{m}_c}{\dot{\theta}_c} = \frac{1}{\sigma - 1} \left( \frac{m_c(1 - m_c)}{\theta_c(1 - \theta_c)} \right) \left[ 1 + \left( \frac{\alpha L}{1 - L} \right) \frac{\theta_c - m_c}{m_c} \right] \equiv f^{m\theta_D}(\theta_c, L, m_c, \sigma). \quad (D.10)$$

Evaluated at any given $(\theta_c, m_c)$ pair, as long as $\sigma > 1$, this slope decreases with $\sigma$ since both the term $\frac{1}{\sigma - 1}$ and $\frac{\alpha L}{1 - L}$ decrease with $\sigma$. The latter follows from Lemma 3 by fixing $\theta_c$. Thus again by the single crossing argument, the equilibrium paths with different $\sigma$ values can cross only once, which is at the steady state, and the equilibrium path becomes flatter as $\sigma$ becomes higher. Consequently, the higher $\sigma$, the higher is $\theta_c^{DS}$.

Further, whenever $m_c = \theta_c$ holds, such as at the steady state, $\dot{m}_c/\dot{\theta}_c = \frac{1}{\sigma - 1}$. If $\sigma = 2$, the projection is the $45^\circ$ line through the origin, and it intersects with the $m_c = \kappa_c$ line at the interior steady state. As we have shown above that the projection becomes flatter with higher $\sigma$, this means that the projection will intersect with $m_c = \kappa_c$ line to the left of the interior steady state if $1 < \sigma < 2$ and to the right of the interior steady state if $\sigma > 2$. That is, $\theta_c^{DS} > \kappa_c$ if and only if $\sigma > 2$.

Evoking the same argument for the clean equilibrium path, we have proven the claims in the proposition.

**E Proof of Lemma 2**

From (32) we conclude that the interior steady state of the simultaneous regime has coordinates $\theta_c = m_c = \kappa_c$; $L = (1 + \rho(\kappa_c; \mu_c)^{-1})/(1 + \alpha) \equiv L^* \equiv L^*$ and is unstable. Consequently, if in the simultaneous regime and $\theta_c \neq \kappa_c$, the economy moves away from the interior...
steady state; if $L > L^k$, $L$ monotonically increases and in finite time necessarily one of the inequalities in (33) is violated; if $L < L^k$, $L$ monotonically decreases and either again one of the inequalities in (33) is violated or $L$ approaches zero and the TVC is violated. Hence the economy cannot stay in the simultaneous regime. This proves part (i).

We prove part (ii) by contradiction. Suppose at time $T$ the economy enters from the clean-only equilibrium the simultaneous regime. We must then have $\lim_{t \to T} \dot{m}_c < 0$ and $\lim_{t \to T} m_c = \kappa_c$. From (30) this requires

$$\theta_c > \left( 1 + \frac{1 - L \mu_c}{\alpha L} \mu_d \right) \kappa_c. \tag{E.11}$$

but according to the second inequality in (33) this contradicts that we enter the simultaneous regime. The proof for the case starting with dirty only is similar.

F Proof of Proposition 3

For the subsequent analysis, we focus on the symmetric case with $\mu_c = \mu_d$. All the results carry through with minor modifications with the two productivity parameters being different. We first characterize the boundaries of regimes in the following lemma.

**Lemma 4.** The projection of the dirty equilibrium path in the $(m_c, L)$ plane lies above the curve $L = \frac{1}{1 + \alpha} \left[ 1 + \frac{\rho}{\mu_d (1 - m_c)} \right]$.

**Proof of Lemma 4.** From the phase diagram, we know $\dot{L}/\dot{m}_c > 0$. Further, the slope of the projection is given by

$$\frac{\dot{L}}{\dot{m}_c} = \frac{L}{1 - m_c} \left[ \frac{\alpha \theta_c L + \rho / \mu_d - [(1 + \alpha) L - 1]}{\alpha \theta_c L - m_c [(1 + \alpha) L - 1]} \right] \equiv f^{LmD}(\theta_c, L, m_c, \sigma). \tag{F.12}$$

Whenever $L = \frac{1}{1 + \alpha} \left[ 1 + \frac{\rho}{\mu_d (1 - m_c)} \right]$ holds, as in the steady state, the term in the square bracket in equation (F.12) becomes one, and $f^{LmD} = \frac{L}{1 - m_c}$, which is obviously a steeper curve than $L$ in the $(m_c, L)$ plane as $1 - m_c \leq 1$. Thus the projection of the equilibrium path can at most intersect with the $L = \frac{1}{1 + \alpha} \left[ 1 + \frac{\rho}{\mu_d (1 - m_c)} \right]$ curve once, which is at the steady state, and for $m_c > 0$, the projection lies above.

We now characterize the level of production labor at which the economy, coming from another regime, enters the dirty-only equilibrium path to the steady state. We denote this level by $L^{DS}$, and the corresponding labor share at this point equals $\theta^{DS}_c$. The counterparts of these “switching points” for the clean-only regime are $L^{CS}$ and $\theta^{CS}_c$. We provide proofs for the dirty-only regime; the clean-only results are similar.
Lemma 5. The level of production labor for which the economy enters the final dirty-only regime never decreases with $\sigma$, i.e. $dL_{DS}(\sigma)/d\sigma \geq 0$.

Proof of Lemma 5. Consider the projection of the dirty equilibrium path in the $(m_c, L)$ plane within the dirty regime. The slope of the projection is given by (F.12). Evaluated at any given $(m_c, L)$ pair, this slope changes with $\sigma$ through changing $\theta_c$. Since

$$\frac{\partial f_{LM}^{mD}}{\partial \theta_c} = \frac{(1 - m_c)\alpha L^2}{(-\dot{m}_c)^2} \left( \frac{\dot{L}}{L} - \frac{\dot{\theta}_c}{1 - m_c} \right)$$

by Lemma 4, $\frac{\partial f_{LM}^{mD}}{\partial \theta_c} \geq 0$. Further, by Lemma 3, for fixed $L$ a higher $\sigma$ translates into a higher $\theta_c$ (see also Figure 12). This proves $dL_{DS}(\sigma)/d\sigma \geq 0$.

Proof of proposition 3. We first check if we can enter the dirty-only regime from the simultaneous regime or the clean only regime. Clearly, such a switch must happen at $m_c = \kappa_c$, with associated values $\theta_c = \theta_{c DS}(\sigma)$ and $L = L_{DS}(\sigma)$. We call this specific triple $(m_c, \theta_c, L)$ the "final regime-switching point". If it does not satisfy (33), a simultaneous regime is not possible and by Lemma 2, the dirty-only regime must be entered from the clean-only regime. Similarly, if the final regime-switching triple $(m_c, \theta_c, L)$ satisfies (33), the dirty-only regime must be entered from the simultaneous regime.

We can write the existence condition for simultaneous regime, (33), as:

$$\dot{m}_c = 0, \theta_c < \kappa_c \Rightarrow L \leq \frac{\kappa_c}{\kappa_c + \alpha(\kappa_c - \theta_c)} \equiv L^{LBS}(\theta_c),$$

$$\dot{m}_c = 0, \theta_c > \kappa_c \Rightarrow L \leq \frac{1 - \kappa_c}{1 - \kappa_c + \alpha(\theta_c - \kappa_c)} \equiv L^{RBS}(\theta_c),$$

where $L^{LBS}$ and $L^{RBS}$ denote the left and right border of the simultaneous regime in the $(\theta_c, L)$- projection, cf the tent-shaped border in Figure 1, panel b, and Figure 13. $L^{RBS}$ is only defined for $\theta_c \geq \kappa_c$, while $L^{LBS}$ only for $\theta_c \leq \kappa_c$.

We define $\bar{\sigma}^{D}$ as the lowest value of $\sigma$ for which the “final-regime switching point” implies $L = 1$:

$$\lim_{\sigma \uparrow \bar{\sigma}^{D}} L^{DS}(\sigma) = 1.$$  

Then for all $\sigma \geq \bar{\sigma}^{D}$ we have $L^{DS} = 1$, since $L^{DS}$ is non-decreasing in $\sigma$ by Lemma 5. Since the slope of the stable path in the $(\theta_c, L)$ plane is given by $\dot{L}/\dot{\theta}_c$, and $\dot{L}/\dot{\theta}_c \to +\infty$ if $\theta_c \to 1$, the equilibrium path must hit the $L = 1$ line before $\theta_c = 1$. In other words, there must exist $2 < \bar{\sigma} < \infty$ such that $L^{DS}(\theta_{c DS}^{D}(\bar{\sigma}^{D})) = 1$. 

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We define $\bar{\sigma}^D$ as the value of $\sigma$ for which the “final-regime switching point” towards the dirty-only regime is located exactly on the right-hand side part of this border:

$$L^{DS}(\bar{\sigma}^D) = L^{RBS}(\theta^{DS}(\bar{\sigma}^D)).$$ (F.17)

We know that at $\sigma = 2$, $\theta^{DS} = \kappa_c$ and $L^{DS} = L^* < 1 = L^{RBS}$. And at $\sigma = \bar{\sigma}^D > 2$, $L^{DS} = 1 > L^{RBS}$, since $L^{DS} = 1$ by definition of $\bar{\sigma}$ while $L^{RBS} < 1$ holds due to monotonicity and the fact that $\theta^{DS}(\bar{\sigma}^D) > \kappa_c$ (due to Lemma 5). Therefore, since the LHS of (F.17) is increasing in $\bar{\sigma}^D > 2$ (by Lemma 5), while the RHS is decreasing (by F.15), as illustrated in Figure 13, $\bar{\sigma}^D \in (2, \bar{\sigma}^D)$ exists.

By Lemma 5 we now have: $\sigma < \bar{\sigma}^D \Rightarrow L^{DS}(\sigma) < L^{RBS}(\theta^{DS}(\sigma))$ so that the final regime-switching point is preceded by the simultaneous regime, which proves Part (ii)a. Similarly, with reversed inequalities, the final regime-switching point is preceded by the clean-only regime. This proves Part (ii)b. And finally, for all $\sigma \geq \bar{\sigma}^D$ we have $L^{DS} = 1$ (i.e. all labor is allocated to production) by Lemma 5. Part (ii)c now follows immediately.

G Modeling Innovation as Creative Destruction

Alternatively, innovation can be modeled as a result of “creative destruction”, where each scientist targets either the clean or the dirty technology, and is then randomly allocated to innovating one machine with a success probability of $\mu_j$, while the size of each quality improvement is denoted by $\gamma$. In this case, equation (7) is replaced by

$$V_{jt} = rV_{jt} - \pi_{jt} + \mu_j s_j V_{jt},$$ (G.18)
where $V_{jt}$ is the value of the patent in technology $j$ in case of successful innovation. The regime separating condition (6) is then replaced by the free entry condition:

$$\mu_j V_{jt} \leq w_t \perp s_{jt} \geq 0.$$  

Comparing the two modeling approaches, one difference is then that reward of successful innovation is evaluated at the marginal value in case of inhouse R&D, while with creative destruction it is evaluated by the total value of the patent.

In terms of forward looking innovation decision making, which technology to target depends on the ratio of the expected patent value in the two sectors $\frac{\mathbb{E}(V_{ct})}{\mathbb{E}(V_{dt})}$. The expectation of higher research effort in one sector on one hand increases the research productivity in that sector due to intra-sectoral spillover, and on the other hand reduces the value of the patent due to creative destruction. However, the latter effect is bounded, as can be seen by considering the extreme case of a one period patent a la Acemoglu et al. (2012). With one period patent, the regime selection condition reduces to

$$\frac{\mathbb{E}(\pi_{ct})}{\mathbb{E}(\pi_{dt})} = \frac{\mu_c}{\mu_d} \left(1 + \gamma \mu_c s_{ct} \right)^{\sigma-2} \left(\frac{Q_{ct-1}}{Q_{dt-1}}\right)^{\sigma-1}$$

(G.20)

where the ratio $\frac{Q_{ct-1}}{Q_{dt-1}}$ captures the effect of history and the ratio $\frac{1+\gamma \mu_c s_{ct}}{1+\gamma \mu_c s_{dt}}$ represents the expectation. Clearly, expectation can also play a role and thus an overlap can only exist if $\sigma > 2$. Further, in this case, an explicit expression of overlap can be derived as follow:

$$(1 + \gamma \eta_{c})^{-\frac{\sigma-2}{\sigma-1}} \left(\frac{\mu_c}{\mu_d}\right)^{-\frac{1}{\sigma-1}} \leq \frac{Q_{ct}}{Q_{dt}} \leq (1 + \gamma \eta_{d})^{-\frac{\sigma-2}{\sigma-1}} \left(\frac{\mu_c}{\mu_d}\right)^{-\frac{1}{\sigma-1}}.$$  

(G.21)

**H Intersectoral Spillover and General Condition for Overlap**

In this appendix we generalize the model in two ways. First, we allow intersectoral spillover of innovation such that

$$q_{ji} = \mu_j s_{ji} Q_j^{1-\nu} Q^{1-\eta},$$

(H.22)

where $Q = Q_c + Q_d$. The model presented in the main text can then be considered a special case where $\eta = 1$. Second, we generalize the final good production to be

$$Y_j = (Q_j^{\epsilon - \nu} L_j)^{1-\alpha} \int_0^1 q_{ji} x_{ji}^\alpha di,$$

(H.23)

where $\epsilon > 0$ can be considered the total input share of the aggregate knowledge or technology stock, where $\nu$ determines how much technology is input or labor augmenting.\textsuperscript{12}

\textsuperscript{12}The exponent $\nu$ is restricted by the requirement that the distribution of productivities across firms affects neither aggregate nor firm-level productivity. This requires $\nu = 1 - \alpha$ in Acemoglu et al. (2012) and $\nu = 1$ in our setting. The specification in both our main text and Acemoglu et al. (2012) implies $\epsilon = \nu$, i.e. there is no independent effect of innovation on productivity otherwise than through intermediate use.
As we want to focus on the role of $\epsilon$ mainly, for analytical simplicity, let $\nu = 1$ in the subsequent analysis. In particular, if $x_{ji}$ is the same across intermediate good firms and proportional to effective labor $Q_j^{\epsilon-1}L_j$, sectoral output will be $Y_j \propto Q_j^\nu L_j$. We thus might also consider $\epsilon$ as a parameter measuring the degree of returns to scale in the model. The model in main text is a special case with $\epsilon = 1$ (constant return to scale). This extension serves to illustrate where the necessary condition $\sigma > 2$ for the existence of overlap comes from.

The intermediate good producer maximizes
\[ H_{ji} = P_{ji}x_{ji} - q_{ji}P_{ji}x_{ji} - ws_{ji} + \lambda_{ji}\mu_j Q^\eta_j Q^{1-\eta}. \]
which leads to
\[ \mu_j\lambda_{ji}Q^\eta_j Q^{1-\eta} - w \leq 0 \perp s_{ji} \geq 0 \]
\[ \lambda_{ji} = r - \frac{1 - \alpha}{\alpha} P_{ji}x_{ji} = r - (1 - \alpha)\frac{1 + \alpha}{\alpha} P_j Q_j^{\epsilon-1}L_j \]
\[ \lambda_{k} = r - \alpha\mu_k L_k \left( \frac{Q_k}{Q} \right)^{\eta-1}, \]
where $k$ is the innovation active sector.

From households’ optimization and market clearing, we further have
\[ C_r = Q_r^{\sigma_r} = Y_r = Q_r^\epsilon L_r \]
\[ \theta_c \equiv \frac{P_c C_c}{PC} = \frac{L_r}{1 + L_r} = \frac{Q_r^{\epsilon(\sigma-1)}}{1 + Q_r^{\epsilon(\sigma-1)}} \]
\[ r = \rho + \hat{P} + \hat{C} = \rho + \hat{w} + \hat{L}. \]

Now define
\[ z_c \equiv \frac{P_r L_r Q_r^{\epsilon-1} Q_r^\eta}{1 + P_r L_r Q_r^{\epsilon-1} Q_r^\eta} = \frac{Q_r^{\epsilon(\sigma-1) + \eta-1}}{1 + Q_r^{\epsilon(\sigma-1) + \eta-1}} \]
\[ m_c \equiv \frac{\lambda_c Q_r^\eta Q^{1-\eta}}{\lambda_c Q_r^\eta Q^{1-\eta} + \lambda_d Q_r^\eta Q^{1-\eta}} = \frac{\lambda_r Q_r^\eta}{1 + \lambda_r Q_r^\eta}. \]

Here, $z_c$ is a predetermined state variable, i.e. a transformation of the relative technology state variable $Q_r$. The transformation ensures that $z_c$ captures all channels through which the initial condition affects the return to innovation. Accordingly, $z_c$ is a sufficient statistic for the allocation of production labor $L_r$, the productivity of labor $Q_r^{\epsilon-1}$, the price of goods $Q_r^{-\epsilon}$, and the productivity of research labor $Q_r^\eta$. 

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We can then derive the differential equation system. In particular, we have
\[ \dot{z}_c = z_c(1 - z_c) (\epsilon(\sigma - 1) + \eta - 1) \tilde{Q}_r \]  
(H.26)
\[ \dot{m}_c = m_c(1 - m_c) (\tilde{\lambda}_r + \eta \tilde{Q}_r), \]  
(H.27)
with \( k \) being the innovation active sector, and \( \tilde{\lambda}_r \propto (m_c - z_c) \) can be easily derived.

The steady states can be derived similarly as in the main text. In particular, we will also have the two corner steady states \((z^d_c, L^d_c, m^d_c) = (0, \frac{1+\rho/\mu}{1+\alpha}, 0)\) and \((z^c_c, L^c_c, m^c_c) = (1, \frac{1+\rho/\mu}{1+\alpha}, 1)\). To see the existence of the overlap, notice that
\[ \frac{\dot{m}_c}{\dot{z}_c} = \frac{1}{\epsilon(\sigma - 1) + \eta - 1} \frac{m_c(1 - m_c)}{z_c(1 - z_c)} \left( \frac{\tilde{\lambda}_r}{\tilde{Q}_r} + \eta \right). \]  
(H.28)
Now, whenever \( m_c = z_c \) holds, as in the corner steady states, \( \tilde{\lambda}_r = 0 \) and
\[ \frac{\dot{m}_c}{\dot{z}_c} = \frac{\eta}{\epsilon(\sigma - 1) + \eta - 1}. \]  
(H.29)
If \( \sigma = 1 + \frac{1}{\epsilon} \), the equilibrium path in each of the corner regimes will be a straight line going through the respective corner steady state and the interior steady state. If \( \sigma > 1 + \frac{1}{\epsilon} \), since \( m_c < z_c \) holds for the dirty equilibrium path (the equilibrium path is to the right of the 45 degree line in the \((z_c, m_c)\) plane) and \( m_c > z_c \) holds for the clean equilibrium path, an overlap of \( z_c \) exists, which translates into an overlap in \( Q_r \). The general condition for the existence of an overlap is thus that the two final goods be strong enough substitutes that the elasticity of substitution \( \sigma \) is larger than one plus the inverse of the degree of returns to scale \( \epsilon \). In other words, the larger the degree of returns to scale, the more likely it is for an overlap to emerge.